Solutions to Hand-In Assignment 2

1. Calculate 3/7 in base 3. [10 pts]

Solution: We are asked to find an infinite series representation for 3/7 of the form

$$\sum_{n=1}^{\infty} \frac{a_n}{3^n},$$

where each a_n is an integer in the set {0, 1, 2}. Since such a series is a sum of nonnegative terms, it is clear that for each index N we must have

$$\sum_{n=1}^{N} \frac{a_n}{3^n} \leq 3/7. \qquad (1)$$

In particular,

$$\frac{a_1}{3} \leq 3/7 < 1,$$

which implies that $a_1 \leq 3(3/7) < 3$. Hence we may let a_1 be the greatest integer less than or equal to 3(3/7). In other words, if we let $f: \mathbf{R} \to \mathbf{Z}$ be the floor function (i.e. the function that rounds down each real number to the greatest integer below this number), then

$$a_1 = f(3(3/7)) = f(9/7) = f(1 + 2/7) = 1.$$

Setting N = 2 in inequality (1), and rearranging, we see that $a_2 \leq 9(3/7) - 3a_1 < 3$. Hence we may let $a_2 = f(9(3/7) - 3a_1) = 0$.
In general, having found $a_1, ..., a_{k-1}$, we may obtain a_k by the recursive formula

$$a_k = f(3^k(3/7) - 3^{k-1}a_1 - ... - 3a_{k-1}).$$

In this fashion, we obtain 3/7 = $0.\overline{102120}$ (base 3). This calculation may be carried out using division in column that you learned in the first grade.

2. Let A = {a, b, c}. Define $F: A \to P(A)$ by

$$F(x) = \begin{cases} \{a,\ b\} & if \quad x = a \\ \{a,\ c\} & if \quad x = b \\ \{b\} & if \quad x = c \end{cases}$$

Compute $S_F = \{x \in A;\ \ x \notin F(x)\}$ [10 pts]

Solution: The only element x that satisfies $x \in F(x)$ is x = a. Hence S_F = {b, c}. Note that this exercise is designed to help you understand the key idea behind Cantor's theorem, which shows that for any set A, the power set P(A) is always bigger.

3. Let A be a proper infinite subset of some set X. If x, y are two distinct elements of X that are not in A, we may set $B = \{x, y\} \cup A$. What is the cardinality of B in terms of the cardinality of A? Justify your answer. [10 pts]

Solution: Since A is infinite, A contains an infinite sequence of distinct elements $\{a_n\}$. Define the function f: A → B by

$$f(u) = \begin{cases} x & if \quad u = a_1 \\ y & if \quad u = a_2 \\ a_n & if \quad u = a_{n+2} \\ u & if \quad u \neq a_n \end{cases}$$

It is easy to see that f is a bijection from A to B. Therefore card(A) = card(B).

4. Find a transfinite number that represents the cardinality of the open interval (0, 1) in terms of $\aleph_0$. Justify your answer. [10 pts]

Solution: Any number x ∈ (0, 1) can be paired with a decimal representation base p. If we fix p = 2 and associate x = $0.a_1a_2...$ (base 2) with the infinite tuple $(a_1, \; a_2, \; ....)$, we see that (0, 1) may thus be identified with the power set of natural numbers P(**N**) (P(**N**) minus some countable set of P(**N**) to be presice). For example, you may regard the tuple (1, 0, 0, 0, 1, 1,…) as the "attendance roaster " of a class, in which infinitely many students 1, 2, 3, … registered, but the ones that showed to class were {1, 5, 6,…}.

By an elementary combinatorial argument, if A is a finite set of cardinality card(A) = n, the power set P(A) has cardinality card(P(A)) = 2^n. With this analogy, we see that card(0, 1) = card(P(**N**)) = $2^{\aleph_0}$. In general, by representing the elements of (0, 1) in base p decimal expansion, we get that card(0, 1) = $p^{\aleph_0}$.

Remark: Cantor showed that "bigger infinities" can be generated by taking power sets. It is interesting to note that it is believed that, given an infinite set A, there does not exist any set, whose cardinality is bigger than card(A) and smaller than card(P(A)). This is the so called continuum hypothesis.